THE UNIVERSAL COVER OF A QUIVER WITH RELATIONS

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Covering techniques in representation theory have become important after the work of Bongartz–Gabriel [2], Gabriel [4] and Riedtmann [9], [10]. The construction of these covers assumes that the Auslander–Reiten quiver is known.

In this paper we associate to any finite-dimensional k-algebra (k algebraically closed) a covering of its usual quiver independent of the Auslander–Reiten quiver (this covering is in fact a particular case of the covering developed by Green [5]).

This construction is defined for any representation type and it is Galois (or regular in the topological sense). Unfortunately, it depends on the choice of an ideal.

We will prove however that it is unique for coverings without oriented cycles and that for the standard case it coincides with the covering given by Gabriel in [4].

We will use this cover to give another characterization of the simply connected algebras considered by Bongartz–Gabriel [2] and Bautista–Larrión–Salmerón [1]. We can use it also to prove that the standard algebras are precisely those algebras of finite representation type having a covering without oriented cycles.

The results proved are essentially contained in the second author's Master thesis [8].

While writing these paper we received from Bretscher and Gabriel [3] a construction which does not require either the knowledge of the indecomposable modules.

J. Waschbüsch has used a similar covering [11], we thank him and F. Larrión for some helpful conversations.

1. The universal cover

Let us fix some notation. Throughout the paper k will be an algebraically closed field. Let Q be a non-necessarily finite quiver with vertices Q_0 and arrows Q_1. For any vertex $x \in Q_0$, x^+ (resp. x^-) will denote the set of vertices y such that there exists an arrow $x \to y$ (resp. $y \to x$). We assume Q has no double arrows and is connected.

Following [2], kQ will be the path category and F the ideal of kQ generated by

the arrows. An ideal of kQ will be called admissible if for any vertex $x \in Q_0$ there exist natural numbers m, n such that

$$F^n(x, \cdot) \subset I(x, \cdot) \subset F^2(x, \cdot) \quad \text{and} \quad F^m(\cdot, x) \subset I(\cdot, x) \subset F^2(\cdot, x).$$

If Q is locally finite and I admissible, $k(Q, I) = kQ/I$ is a locally bounded category. Conversely, for any locally bounded k-category $\mathscr{C}$ there is a locally finite quiver Q and an admissible ideal I such that $\mathscr{C} \cong k(Q, I)$. (See [2].)

A pair (Q, I) as above will be called a quiver with relations.

1.1. Definitions. A group G of (Q, I)-automorphisms is a group of automorphisms of the quiver Q which preserve the ideal I. The group is called *admissible* if for each vertex $x \in Q_0$ the G-orbit contains at most one vertex from x^+ and one from x^-.

Given an admissible group G of (Q, I)-automorphisms, we can define the *orbit quiver* Q/G. Let $\pi : Q \to Q/G$ be the natural map and denote $\bar{I} = \pi(I)$ the ideal of Q/G induced by I. In this situation we have the following lemma:

1.2. Lemma. $(Q/G, \bar{I})$ *is a quiver with relations. The natural map* $\pi : (Q, I) \to (Q/G, \bar{I})$ *is a surjective morphism of quivers with relations such that for every* $x \in Q_0$, *the induced functions* $x^+ \to \pi(x)^-$ *and* $x^- \to \pi(x)^-$ *are bijective.* $\square$

A morphism of quivers with relations $f : (\Delta, J) \to (Q, I)$ is a *covering* if there exists an admissible group of $(\Delta, J), G$ such that the following diagram commutes.

By 1.2, covering maps have the unique lifting of walks property. We ask for the relations which can be lifted by a covering map. The following concept gives the answer, it will be very useful throughout the paper.

1.3. Definition. Let (Q, I) be a quiver with relations. A relation $\varrho = \sum_{i=1}^{n} \lambda_i u_i \in I(x, y)$ with $\lambda_i \in k^*$ and u_i a directed path from x to y, is a *minimal* relation if $n \geq 2$ and for every non-empty proper subset K of $\{1, \ldots, n\}$, $\sum_{i \in K} \lambda_i u_i \notin I(x, y)$.

As usual, ϱ will be called a *zero* relation when $n = 1$.

Of course, every relation is sum of minimal and zero relations.

1.4. Proposition. *Let* $f : (\Delta, J) \to (Q, I)$ *be a covering morphism, defined by the action of* G. *Let* $\varrho \in I(x, y)$ *be a minimal relation and let* $\bar{x} \in \Delta_0$ *with* $f(\bar{x}) = x$. *Then, there exists* $\bar{y} \in \Delta_0$ *and* $\bar{\varrho} \in J(\bar{x}, \bar{y})$ *with* $f(\bar{\varrho}) = \varrho$.

Proof. We write $\varrho = \sum_{i=1}^{n} \lambda_i u_i$ with $\lambda_i \in k^*$ and u_i, $i = 1, \ldots, n$ pairwise different directed paths (we say ϱ is in a reduced form). Since f is a covering map, there exist $\varrho_i \in J(x_i, y_i)$, $i = 1, \ldots, m$ and $\varrho = \sum_{i=1}^{m} f(\varrho_i)$. As $f(\bar{x}) = x = f(x_i)$, there is a $g_i \in G$ with $g_i x_i = \bar{x}$ for $i = 1, \ldots, m$. Then, $\sigma_i = g_i f_i \in J(\bar{x}, g_i y_i)$ and $\varrho = \sum_{i=1}^{m} f(\sigma_i)$. We can clearly assume the vertices $g_i y_i$ are pairwise different. We write $\sigma_i = \sum_{j=1}^{n_i} \mu_{ij} v_{ij}$ in reduced form. So we obtain

$$\sum_{i=1}^{n} \lambda_i u_i = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \mu_{ij} f(v_{ij})$$

which is also in reduced form, because of the unique path lifting property.
Therefore, there exists a bijection

$$\tau : \{1, \ldots, n\} \to \{(l, j) \mid l \in \{1, \ldots, m\}, j \in \{1, \ldots, n_l\}\}$$

such that $u_i = f(v_{\tau(i)})$ and $\lambda_i = \mu_{\tau(i)}$ for every $i \in \{1, \ldots, n\}$.
If we had $m > 1$, $\emptyset \neq K = \tau^{-1}\{(1, j) \mid j = 1, \ldots, n_1\} \subsetneq \{1, \ldots, n\}$ such that

$$\sum_{i \in K} \lambda_i u_i = \sum_{j=1}^{n_1} \mu_{1j} f(v_{1j}) = f(\sigma_1) \in f(J) = I,$$

which is a contradiction in case ϱ is a minimal or zero relation. Thus $m = 1$ and $\varrho = f(\sigma_1)$. □

Denote now by $m(I)$ the set of minimal relations of the ideal I. By 1.4, the category of coverings of (Q, I) with covering maps is just the category $\mathrm{Cov}(Q, m(I))$ defined in [5].
Applying [5; (1.2)] to our particular case we get

1.5. Corollary. *There exists a universal cover of* (Q, I). *That is, there is a covering map* $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ *such that for any other covering map* $f : (\bar{Q}, \bar{I}) \to (Q, I)$ *there exists a covering* $\pi' : (\tilde{Q}, \tilde{I}) \to (\bar{Q}, \bar{I})$ *with* $f\pi' = \pi$. *If* $\tilde{x} \in \tilde{Q}_0$ *and* $\bar{x} \in \bar{Q}_0$ *are such that* $\pi\tilde{x} = f\bar{x}$, *then* π' *can be uniquely chosen so that* $\pi'\tilde{x} = \bar{x}$.

We give here a brief description of $(\tilde{Q}, \tilde{I})$, because it will be required later.
Fix a vertex $x_0 \in Q_0$. Let W be the topological universal cover of Q with base point in x_0. There is a natural map $p : W \to Q$ given by the action of the fundamental group $\Pi_1(Q, x_0)$. Let $N(Q, m(I), x_0)$ be the subgroup of $\Pi_1(Q, x_0)$ defined in [5]. That is, $N(Q, m(I), x_0)$ is the normal subgroup of $\Pi_1(Q, x_0)$ generated by all elements of the form $[y^{-1} u^{-1} vy]$ where y is a path from x_0 to x and u, v are directed paths from x to y such that there is an element $\sum_{i=1}^{n} \lambda_i w_i \in m(I)$ with $u = w_1$ and $v = w_2$.
Then, $\tilde{Q}$ is the orbit quiver $W/N(Q, m(I), x_0)$ and the map $\pi : \tilde{Q} \to Q$ is given by the action of the group $\Pi_1(Q, I) = \Pi_1(Q, x_0)/N(Q, m(I), x_0)$. The relations $m(I)$ can be lifted to $\tilde{Q}$. So, $\tilde{I}$ is the admissible ideal of $k\tilde{Q}$ generated by the liftings of the elements of $m(I)$ and zero relations in I.

Given a basic and indecomposable k-algebra A, we can build a quiver with relations (Q, I) such that $A \cong k(Q, I)$. We can now obtain the universal cover $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ and induce a functor $k(\pi) : k(\tilde{Q}, \tilde{I}) \to A$. We would like to say that $\tilde{A} = k(\tilde{Q}, \tilde{I})$ is the universal cover of A. Unfortunately this cannot be done because of the ambiguity in the choice of the ideal I. Let's remind the following example given by Riedtmann:

$$\mathcal{V}: \xleftarrow[\quad]{\gamma} \cdot 1 \underset{\gamma}{\overset{\beta}{\rightleftharpoons}} \cdot 2$$

and let I_1 be the ideal generated by $\alpha^2 - \gamma\beta$, $\beta\gamma - \beta\alpha\gamma, \alpha^4$ and I_2 generated by $\alpha^2 - \gamma\beta, \beta\gamma, \alpha^4$.

Let $A_1 = k(Q, I_1)$ and $A_2 = k(Q, I_2)$ be the corresponding algebras. If char $k \neq 2$, $A_1 \cong A_2$. Nevertheless, the universal cover of (Q, I_1) is itself and the universal cover of (Q, I_2) is an infinite quiver.

2. Universal covers without oriented cycles

Let (Q, I) be a quiver with relations. Throughout this section $A = k(Q, I)$ will be a locally representation-finite category (see [2]).

We are interested in the existence of coverings of (Q, I) without oriented cycles. Clearly, this is equivalent to ask for the same condition on the universal cover.

We will use the following conditions proved by Jans [6].

Let P_1 and P_2 be two indecomposable projective A-modules. Then $\operatorname{End}_1(P_1)$ and $\operatorname{End}_1(P_2)$ are uniserial rings and $\operatorname{Hom}_1(P_1, P_2)$ is a $\operatorname{End}_1(P_2)$-$\operatorname{End}_1(P_1)$ bimodule which is uniserial either as $\operatorname{End}_1(P_2)$-module or as $\operatorname{End}_1(P_1)$-module.

2.1. Proposition (Gabriel [4]). *Let* $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ *be a covering map. Then* $k(\tilde{Q}, \tilde{I})$ *is also a locally representation-finite category.*

We recall that a category ℓ is called Schurian if for any two objects $x, y \in \operatorname{Ob} \ell$, $\dim_k \ell(x, y) \leq 1$. We say that (Q, I) is Schurian when $k(Q, I)$ is so.

2.2. Lemma. *Assume Q without oriented cycles. Then (Q, I) is Schurian.*

Proof. This is a direct consequence of the results of Jans [6].

Let $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ be the universal cover of (Q, I).

2.3. Lemma. *Assume $\tilde{Q}$ has no oriented cycles. Then:*

(D) *If* $\sum_i \lambda_i u_i \in I$ *is a minimal relation, then for every two different* $i, j \in \{1, \ldots, n\}$ *there exists* $c \in k^*$ *such that* $u_i + cu_j \in I$.

(C) If $x \xrightarrow{u} y \underset{w}{\overset{v}{\rightrightarrows}} z$ are directed paths with $vu, wu \notin I$ and $\lambda \in k^*$, then we have $vu + \lambda wu \in I$ if and only if $v + \lambda w \in I$.

Proof. By 2.1 and 2.2, $(\tilde{Q}, \tilde{I})$ is Schurian.

(D) Let $\bar{x} \in \tilde{Q}_0$ with $\pi \bar{x} = x$. Take liftings $\bar{u}_i$ of u_i starting at $\bar{x}$, $i = 1, \ldots, n$. Then $e(\bar{u}_i) = e(\bar{u}_j)$, $i, j \in \{1, \ldots, n\}$. As $\bar{u}_i, \bar{u}_j \notin \tilde{I}(\bar{x}, e(\bar{u}_i))$, there exists $c \in k^*$ with $\bar{u}_i + c\bar{u}_j \in \tilde{I}(\bar{x}, e(\bar{u}_i))$. Applying $\pi, u_i + cu_j \in I(x, e(u_i))$.

(C) As $vu + wu \in I$ is a minimal relation, $v \sim w$ and the result follows as in the proof of (D). $\square$

Properties (C) and (D) are intrinsic to the quiver with relations (Q, I) and will be useful.

2.4. Proposition. *Assume (Q, I) satisfies* (D). *Let $x, y \in Q_0$ such that $k(Q, I)(x, y)$ is uniserial as $k(Q, I)(x, x)$ module. Then, there exists a directed walk form x to y and a directed cycle at x such that for any directed walk v from x to y there is $\lambda \in k^*$ and $n \in \mathbb{N}$ with $v + \lambda u w^n \in I(x, y)$.*

Proof. We write $A_x := k(Q, I)(x, x)$ and $R_x = \text{rad } A_x$.

As $k(Q, I)(x, y) \supsetneq \text{rad } A_x k(Q, I)(x, y)$, we take an element f in the difference. We can write $f = \sum_i^n \lambda_i \bar{u}_i$, where u_i is a directed path from x to y. Then, there is a directed path u from x to y with $\bar{u} \notin \text{rad } A_x k(Q, I)(x, y)$.

As A_x is uniserial, the radical series is a composition series. If $R_x = 0$, then $k(Q, I)(x, y)$ is simple and the result trivial. Assume $R_x \neq 0$, we choose a directed cycle w at x with $\bar{w} \notin R_x^2$. It is easy to check that $A_x = \sum_{i=0}^m kz^i$, where $z = \bar{w} \in A_x$ for some $m \in \mathbb{N}$.

Take now an arbitrary directed walk v from x to y. As $k(Q, I)(x, y)$ is A_x uniserial module, $\bar{u} A_x \subseteq \bar{v} A_x$ or $\bar{v} A_x \subseteq \bar{u} A_x$. Let's see what happens when $\bar{u} A_x \subseteq \bar{v} A_x$. There are scalars $\lambda_i \in k$, $i = 0, \ldots, m$ such that

$$\bar{u} = \sum_{i=0}^m \lambda_i \bar{v} z^i = \sum_{i=0}^m \lambda_i \overline{vw^i}.$$

That is, $u - \sum_{i=0}^m \lambda_i v w^i \in I(x, y)$ can be expressed as sum of zero and minimal relations. But $u \notin I(x, y)$, so we have some $\emptyset \neq K \subseteq \{0, \ldots, m\}$ satisfying $u - \sum_{i \in K} \lambda_i v w^i \in I(x, y)$ is a minimal relation. As we are assuming (D), there is $c \in k^*$ and $n \in \{0, \ldots, m\}$ with $u + cvw^n \in I(x, y)$.

Then, $c\bar{v}z^n = \bar{u} \notin \text{rad } A_x k(Q, I)(x \ y) = k(Q, I)(x, y) \cdot R_x$ and having $z \in R_x$, we deduce $n = 0$. So, $v + c^{-1} u \in I(x, y)$. The inclusion $\bar{v} A_x \subseteq \bar{u} A_x$ by the same process would imply the result. $\square$

2.5. Proposition. *Assume (Q, I) satisfies* (C) *and* (D). *Let $x_i \xrightarrow{\alpha_i} x_{i+1}$ be an arrow in Q and $\mu = \alpha_n \cdots \alpha_1$ a directed walk with $\mu \notin I$. Suppose $\mu' = \varrho_{n+1} \alpha_n \varrho_n \cdots \varrho_2 \alpha_1 \varrho_1$ such that ϱ_i is a directed cycled in x_i, for $i = 1, \ldots, n$, and $\bar{\mu}' = c\bar{\mu}$ in $k(Q, I)$ with $c \in k^*$. Then all the walk ϱ_i are trivial.*

Proof. We can assume $k(Q, I)(x_1, x_{n+1})$ is a Λ_{x_1} uniserial module, where $\Lambda_x :=$ $k(Q, I)(x, x)$. Define

$$\mu'' := \alpha_n \varrho_n \alpha_{n-1} \cdots \varrho_2 \alpha_1 \varrho_1 \notin I(x_1, x_{n+1}),$$

we must have $\bar{\mu}' \Lambda_{x_1} \subset \bar{\mu} \Lambda_{x_1}$ or $\bar{\mu} \Lambda_{x_1} \subset \bar{\mu}'' \Lambda_{x_1}$. We prove $\bar{\mu}'' \Lambda_{x_1} \subset \bar{\mu} \Lambda_{x_1}$.

Let w be a directed cycle at x_1 such that $\{\bar{w}^i \mid i = 0, \ldots, m_{x_1}\}$ generates Λ_{x_1} as k-vector space. Then there is a scalar $\lambda \in k^*$ and $m \in \mathbb{N}$ with $\mu + \lambda \mu'' w^m \in I$, because of (D).

Induction on n. If $n = 1$, $\alpha_1 + \lambda \alpha_1 \varrho_1 w^m \in I$ and since I is admissible, ϱ_1 is trivial and $m = 0$. Then, $\mu'' + \lambda^{-1} \mu \in I$.

Assume $n > 1$, by (C),

$$\alpha_{n-1} \cdots \alpha_1 + \lambda \varrho_n \alpha_{n-1} \varrho_{n-1} \cdots \varrho_2 \alpha_1 \varrho_1 w^m \in I.$$

Suppose first $k(Q, I)(x_1, x_n)$ is Λ_{x_1} uniserial, then by the induction hypothesis there exist $\lambda' \in k^*$ and $t \in \mathbb{N}$ with $\alpha_{n-1} \varrho_{n-1} \cdots \varrho_2 \alpha_1 \varrho_1 + \lambda' \alpha_{n-1} \cdots \alpha_1 w^t \in I$. Multiplying by ϱ_n and w^m, we get

$$\lambda \varrho_n \alpha_{n-1} \varrho_{n-1} \cdots \varrho_2 \alpha_1 \varrho_1 w^m + \lambda'' \varrho_n \alpha_{n-1} \cdots \alpha_1 w^{m+t} \in I$$

and finally,

$$\alpha_{n-1} \cdots \alpha_1 + \kappa_1 \varrho_n \alpha_{n-1} \cdots \alpha_1 w^{m+t} \in I \quad \text{with } \kappa_1 \in k^*.$$

Using the nilpotency of the elements of Λ_{x_1} and Λ_{x_n} we conclude that ϱ_n and w^{m+t} are trivial, so $m = 0$, and $\mu + \lambda'^{-1} \mu'' \in I$.

In the case $k(Q, I)(x_1, x_n)$ is Λ_{x_n} uniserial. Also by induction hypothesis – observe that μ'' is defined by cancelling the cycle opposite to the uniserial extreme – there exist $\lambda' \in k^*$ and $t \in \mathbb{N}$ with $\varrho_n \alpha_{n-1} \varrho_{n-1} \cdots \varrho_2 \alpha_1 + \lambda' w^t \alpha_{n-1} \cdots \alpha_1 \in I$ and we proceed exactly as before getting $\mu + \lambda'^{-1} \mu'' \in I$.

So we have proved $\bar{\mu}'' \Lambda_{x_1} \subset \bar{\mu} \Lambda_{x_1}$. So, $\mu'' + \lambda \mu w^m \in I(x_1, x_{n+1})$ for some $\lambda \in k^*$ and $m \in \mathbb{N}$. We have

$$\varrho_{n+1} \alpha_n \varrho_n \cdots \varrho_2 \alpha_1 \varrho_1 + c \alpha_n \cdots \alpha_1 \in I(x_1, x_{n+1}),$$

$$\alpha_n \varrho_n \cdots \varrho_2 \alpha_1 \varrho_1 + \lambda \alpha_n \cdots \alpha_1 w^m \in I(x_1, x_{n+1}).$$

As before, $\alpha_n \cdots \alpha_1 + \lambda' \varrho_{n+1} \alpha_n \cdots \alpha_1 w^m \in I(x_1, x_{n+1})$ with $\lambda' \in k^*$. So $m = 0$ and ϱ_{n+1} is trivial. Then by (C), $\varrho_n \alpha_{n-1} \cdots \varrho_2 \alpha_1 \varrho_1 + c \alpha_{n-1} \cdots \alpha_1 \in I(x_1, x_n)$ and by induction hypothesis $\varrho_1, \ldots, \varrho_n$ are trivial.

We return to the problem of how the universal cover depends on the chosen ideal.

2.6. Proposition. *Assume $(Q, I_1), (Q, I_2)$ are locally representation finite quivers with relations. Such that $k(Q, I_1) \cong k(Q, I_2)$ and (Q, I_1) and (Q, I_2) satisfy (C), (D). Let $\pi_i : (\tilde{Q}_i, \tilde{I}_i) \to (Q, I_i)$ be the universal cover $i = 1, 2$, then $\tilde{Q}_1 = \tilde{Q}_2$ and $\Pi_1(Q, I_1) = \Pi_1(Q, I_2)$.*

Proof. For both affirmations we only need to show that two paths belonging to a minimal relation in I_1 also belong to a minimal relation in I_2.

Suppose $0 \to I_1 \to kQ \to \mathscr{C} \to 0$ and $0 \to I_2 \to kQ \to \mathscr{C} \to 0$ are exact. Given $\alpha \in Q_1$, we can take $f(\alpha) \in kQ$ with $\psi(f(x)) = \phi(\alpha)$ in $\mathscr{C}$ and extend f to a morphism of categories such that

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_1 & \longrightarrow & kQ & \overset{\phi}{\longrightarrow} & \mathscr{C} & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow f & & \| & & \\
0 & \longrightarrow & I_2 & \longrightarrow & kQ & \overset{\psi}{\longrightarrow} & \mathscr{C} & \longrightarrow & 0 \\
 & & \downarrow g & & \downarrow f & & \| & & \\
0 & \longrightarrow & I_1 & \longrightarrow & kQ & \overset{\phi}{\longrightarrow} & \mathscr{C} & \longrightarrow & 0
\end{array}
$$

is exact and commutative.

Let μ be a directed path in Q, $\mu \in I_1$. We write $\mu = \alpha_n \cdots \alpha_1$ with α_i an arrow. By 2.4, without lost of generality, there is a cycle in the starting point of α_1, w_1 such that

$$
f(\alpha_1) + \lambda_1 \alpha_1 + \sum_{i=1}^{n_1} \kappa_{1i} \alpha_1 w_i' \in I_2,
$$

and $\lambda_1 \in k^*$. Doing this for each arrow we obtain

$$
f(\alpha_n) \cdots f(\alpha_1) = f(\mu) \in I_2
$$

and

$$
f(\mu) + \lambda \mu + \sum_{\substack{\mu = \mu_1, \ldots, \mu_l \\ l > 1}} \kappa_{\mu_1, \ldots, \mu_l} \varrho_{l+1} \mu_l \varrho_l \cdots \mu_2 \varrho_2 \mu_1 \varrho_1 \in I_2
$$

with $\lambda \in k^*$ and ϱ_i a non-trivial directed cycle in the starting vertex of μ_i, $i = 2, \ldots, l$. If we had $\mu \notin I_2$, then by (D) we would have a non-trivial partition $\mu = \mu_l \cdots \mu_1$ and $\mu + c\varrho_{l+1} \mu_l \varrho_l \cdots \mu_2 \varrho_2 \mu_1 \varrho_1 \in I_2$, $c \in k^*$. But this contradicts 2.5, so $\mu \in I_2$.

Take now $c\mu + v \in I_1$ a minimal relation – by (D) it is enough to prove the result for this kind of relations. As before we get $f(c\mu + v) \in I_2$ and

$$
f(c\mu + v) + \lambda \mu + \lambda' v + \sum_{\substack{\mu = \mu_1, \ldots, \mu_l \\ l > 1}} \kappa_{\mu_1, \ldots, \mu_l} \varrho_{l+1} \mu_l \cdots \mu_1 \varrho_1
$$

$$
+ \sum_{\substack{v_1, \ldots, v_l = v \\ l > 1}} \kappa'_{v_1, \ldots, v_l} \varrho'_{l+1} v_l \cdots v_1 \varrho'_1 \in I_2
$$

with λ, $\lambda' \in k^*$ and ϱ_i, ϱ'_i directed cycles.

As $\mu \notin I_1$, then $\mu \notin I_2$. If μ and v form no part of a minimal relation in I_2 as we have (D) and 2.5, there must be a non-trivial partition $v = v_l, \ldots, v_1$ and $\kappa \in k^*$ with $\mu + \kappa \varrho'_{l+1} v_l \cdots v_1 \varrho_1 \in I_2$ a minimal relation.

Applying g to this relation, we get

$$d\mu + \sum_{\substack{u\ u,\ \ldots,u_i \\ i\ 1}} c_{u_1,\ldots,u_i}\varrho_{t+1}\mu_t\cdots\mu_1\varrho_1 + \sum_{\substack{v \\ v_i,\ldots,v_j}} c_{v_1,\ldots,v_j}\varrho''_{t+1}v''_t\cdots v'_1\varrho''_1 \in I_1$$

with $d \in k^*$. Again by 2.5 we must have $v = v'_t\cdots v'_1$, $t > 1$ and $\kappa' \in k^*$ with

$$\mu + \kappa'\mu''_{t+1}v'_t\varrho''_t\cdots v'_1\varrho''_1 \in I_1.$$

But we also i ad $c\mu + v \in I_1$, so we obtain $c^{-1}v + \kappa'\varrho''_{t+1}v'_t\varrho''_t\cdots v'_1\varrho''_1 \in I_1$ which contradicts 2.5. .hen there is a $c' \in k^*$ and $\mu + c'v \in I_2$. $\square$

Observe that in Proposition 2.6 above $\lambda\mu + \lambda'v \in I_2$. This will be important in the next result which is the main one of the section.

2.7. Theorem. *Assume $(Q, I_1), (Q, I_2)$ are locally representation-finite quivers with relations, such $k(Q, I_1) \cong t \cong k(Q, I_2)$ and $(Q, I_1), (Q, I_2)$ satisfy* (C) *and* (D). *Let $\pi_i: (\tilde{Q}_i, \tilde{I}_i) \to (Q, I_i)$ be a universal cover $i = 1, 2$. Then there exist isomorphisms $h: k(Q, I_1) \to k(Q, I_2)$ and $h: k(\tilde{Q}_1, \tilde{I}_1) \to k(\tilde{Q}_2, \tilde{I}_2)$ making the following square commutative:*

$$
\begin{array}{ccc}
k(\tilde{Q}_1, \tilde{I}_1) & \xrightarrow{\;\tilde{h}\;} & k(\tilde{Q}_2, \tilde{I}_2) \\
{\scriptstyle k(\pi_1)}\downarrow & & \downarrow{\scriptstyle k(\pi_2)} \\
k(Q, I_1) & \xrightarrow{\;h\;} & k(Q, I_2)
\end{array}
$$

Proof. With the notation of 2.6 we write the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \to & I_1 & \to & kQ & \xrightarrow{\varphi} & t & \to & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\
0 & \to & I_2 & \to & kQ & \xrightarrow{\psi} & t & \to & 0 \\
 & & {\scriptstyle g}\downarrow & & {\scriptstyle g}\downarrow & & \| & & \\
0 & \to & I_1 & \to & kQ & \xrightarrow{\varphi} & t & \to & 0
\end{array}
$$

Let $x \xrightarrow{\;\alpha\;} y$ be an arrow in Q, as in 2.6 $f(\alpha) + \lambda_\alpha\alpha + \sum_{i}^{m} \kappa_i\alpha w' \in I_2(x, y)$ for some $\lambda_\alpha \in k^*$. We define $h: kQ \to kQ$ as $h(\alpha) = \lambda_\alpha\alpha$, which is an isomorphism of categories. In 2.6 and the observation above we showed that h preserves zero and minimal relations. Then, $h(I_1) \subset I_2$ and h extends to a morphism of categories $h: k(Q, I_1) \to k(Q, I_2)$. To prove it is an isomorphism, it will be enough to show that its inverse also preserves relations.

Again, for $x\xrightarrow{\alpha} y$ an arrow in Q, $f(\alpha)=\lambda_\alpha\alpha+r_\alpha$ with $r_\alpha\in F^2(x,y)$. Similarly, $g(\alpha)=\lambda'_\alpha\alpha+r'_\alpha$ with $\lambda'_\alpha\in k^*$ and $r'_\alpha\in F^2(x,y)$. Therefore, $gf(\alpha)=\lambda_\alpha\lambda'_\alpha\alpha+r''_\alpha$ with $r''_\alpha\in F^2(a,y)$ and $\phi(\alpha)=\phi gf(\alpha)=\lambda_\alpha\lambda'_\alpha\phi(\alpha)+\phi(r''_\alpha)$ with $\phi(r''_\alpha)\in\mathrm{rad}^2\ \mathscr{C}(x,y)$. So $\lambda_\alpha\lambda'_\alpha=1$, and $t:kQ\to kQ$, $t(\alpha)=\lambda'_\alpha\alpha$ is inverse of h and $t(I_2)\subset I_1$. In conclusion, $h:k(Q,I_1)\to k(Q,I_2)$ is an isomorphism.

By 2.6, $\tilde{Q}_1=\tilde{Q}_2$. Define $\tilde{h}:k\tilde{Q}_1\to k\tilde{Q}_2$ as the identity in the vertices, and for $x\xrightarrow{\alpha} y$ arrow in $\tilde{Q}_1$, $\tilde{h}(\alpha)=\lambda_{\pi_1\alpha}\alpha$. By what we observed before, it is enough to prove $\tilde{h}(\tilde{I}_1)\subset\tilde{I}_2$. But this is trivial because $\pi_1=\pi_2$ as quiver morphisms and $\tilde{h}(\alpha)$ is just a lifting of $h\pi_1\alpha$.

It is also clear that $k(\pi_2)\tilde{h}=hk(\pi_1)$. ☐

In particular 2.7 is valid when $\tilde{Q}_1$ and $\tilde{Q}_2$ have no oriented cycles, by Lemma 2.3.

3. The universal cover of a standard category

In this part of the work we will study the relations between the universal cover of the Auslander–Reiten quiver and the universal cover of the ordinary quiver of a locally representation-finite category.

We will see that the construction of the universal cover we gave in Section 1 coincides in some important cases with the following given by Gabriel in [4]: let N be a locally representation finite category and $f:k(\tilde{\Gamma}_N)\to\mathrm{ind}\,N$ a covering functor associated to the universal cover $\tilde{\Gamma}_N$ of the Auslander–Reiten quiver of N,Γ_N, we denote by M the full subcategory of the projective vertices of $k(\tilde{\Gamma}_N)$ in such a way that the restriction of $F,F':M\to N$ is a covering functor. The category M is called the universal cover of N.

3.1. Definition [2]. A locally finite-dimensional k-category $\mathscr{C}$ is called *square free* if the vector spaces $r\mathscr{C}(a,b)/r^2\mathscr{C}(a,b)$ have dimension smaller or equal to 1 over k, for every pair $a,b\in\mathrm{Ob}\,\mathscr{C}$, where $r\mathscr{C}$ denotes the radical of the category $\mathscr{C}$.

If $F:\mathscr{C}\to\mathscr{D}$ is a covering functor and $\mathscr{D}$ is square free, clearly $\mathscr{C}$ is also so.

3.2. Definition. Let $F:\mathscr{C}\to\mathscr{D}$ be a covering functor. We denote by $\mathscr{A}(F)$ the group of $\mathscr{C}$-automorphisms which preserve F.

A group of equivalences of $\mathscr{C}$ *acts freely* on arrows of $\mathscr{C}$ if for any $g\in\mathscr{A}(F)$ with $gx=x$ for some $x\in\mathrm{Ob}\,\mathscr{C}$, then $g/\mathrm{Ob}\,\mathscr{C}=\mathrm{id}$ and for every $\alpha\in r\mathscr{C}(y,z)$ such that $0\neq\bar{\alpha}\in r\mathscr{C}(y,z)/r^2\mathscr{C}(y,z)$ we have $g(\alpha)=\alpha$.

Recall that for a locally bounded category $\mathscr{C}$ there is associated a locally finite quiver Q and $\mathscr{C}$ is a connected category if and only if Q is connected.

3.3. Proposition. *Let $F:\mathscr{C}\to\mathscr{D}$ be a covering functor between locally bounded square free categories; assume $\mathscr{C}$ connected. Then, $\mathscr{A}(F)$ acts freely on arrows of $\mathscr{C}$.*

Proof. Let $g \in \mathscr{A}(F)$ with $gx = x$ for some $x \in \text{Ob } \mathscr{C}$. Let Q be the quiver associated to $\mathscr{C}$, and $y \to x$ an arrow there. So we have $0 \neq r\mathscr{C}(y, x)/r^2\mathscr{C}(y, x)$. Suppose that $gy \neq y$; as $gx = x$ and g is an equivalence of $\mathscr{C}$, we also have $0 \neq r\mathscr{C}(gy, x)/r^2\mathscr{C}(gy, x)$. But since F is a covering functor between locally bounded categories, we have by [2],

$$r\mathscr{C}(y, x)/r^2\mathscr{C}(y, x) \oplus r\mathscr{C}(gy, x)/r^2\mathscr{C}(gy, x) \hookrightarrow \bigoplus_{Fz = Fy} r\mathscr{C}(z, x)/r^2\mathscr{C}(z, x)$$

$$\cong r\mathscr{D}(Fy, Fx)/r^2\mathscr{D}(Fy, Fx)$$

but this contradicts the fact that the last space has dimension at most 1. So $gy = y$ and since Q is connected, $g/\text{Ob } \mathscr{C} = \text{id}$.

Take now $\alpha \in r\mathscr{C}(z, y)$ such that $0 \neq \bar{\alpha} \in r\mathscr{C}(z, y)/r^2\mathscr{C}(z, y)$. As g is the identity on objects, $0 \neq \overline{g\alpha} \in r\mathscr{C}(z, y)/r^2\mathscr{C}(z, y)$ which has dimension 1 over k. So there are $\lambda \in k^*$ and $m \in r^2\mathscr{C}(z, y)$ with $g\alpha = \lambda\alpha + m$. Applying F we get, $F\alpha = Fg\alpha = \lambda F\alpha + Fm$. As F is a covering functor,

$$(1 - \lambda)F\alpha = Fm \in r^2\mathscr{D}(Fz, Fy) \quad \text{and} \quad 0 \neq F\alpha \in r\mathscr{D}(Fz, Fy)/r^2\mathscr{D}(Fz, Fy).$$

So $\lambda = 1$, $Fm = 0$. Therefore, $m = 0$ and $g\alpha = \alpha$. $\square$

3.4. Proposition. *Let $F : \mathscr{C} \to \mathscr{D}$ be a covering functor between locally bounded square free categories. Assume $\mathscr{C}$ connected and $\mathscr{A}(F)$ acts transitively on fibres of objects. Let $\bar{Q}$ be the quiver associated to $\mathscr{C}$ and Q the associated to $\mathscr{D}$.*

Then, there exists a quiver morphism $\pi : \bar{Q} \to Q$ and two functors $r : k\bar{Q} \to \mathscr{C}$ and $\varrho : kQ \to \mathscr{D}$ such that

is exact and commutative.

Proof. We define $\text{Irr } \mathscr{C} \subset \text{Ob } \mathscr{C} \times \text{Ob } \mathscr{C}$ such that $(x, y) \in \text{Irr } \mathscr{C}$ if and only if $r\mathscr{C}(x, y)/r^2\mathscr{C}(x, y) \neq 0$. Observe that $\mathscr{A}(F)$ acts on $\text{Irr } \mathscr{C}$ and induces a partition of it. For each class θ of this partition we select a representative $(x, y) \in \theta$ and $\alpha_{(x, y)} \in r\mathscr{C}(x, y)$ with $0 \neq \bar{\alpha}_{(x, y)} \in r\mathscr{C}(x, y)/r^2\mathscr{C}(x, y)$. For $g \in \mathscr{A}(F)$ we put $\alpha_{(gx, gy)} = g\alpha_{(x, y)}$. As $\mathscr{A}(F)$ acts freely on arrows of $\mathscr{C}$ by 3.3, $\alpha_{(x, y)}$ is a well-defined irreducible for each $(x, y) \in \text{Irr } \mathscr{C}$.

As $\mathscr{C}$ is square free, $r : k\bar{Q} \to \mathscr{C}$ such that $r(x \to y) = \alpha_{(x, y)}$ is a well-defined, full and dense functor. As F is a covering functor all the arrows in Q are of the form $Fx \to Fy$ with $(x, y) \in \text{Irr } \mathscr{C}$, so we define $\varrho : kQ \to \mathscr{D}$ by $\varrho(Fx \to Fy) = F\alpha_{(x, y)}$. We must prove this definition does not depend on the choice of (x, y). So we assume we have another $(x', y') \in \text{Irr } \mathscr{C}$ giving the same arrow in Q. By assumption, there is some

$g \in \mathscr{G}(F)$ with $gx = x$. Then, $(gx, gy) \in \mathrm{Irr}\, \mathscr{C}$ also covers $Fx \to Fy$. So, $gy = y'$, and $F\alpha_{(x', y')} = F\alpha_{(gx, gy)} = Fg\alpha_{(x, y)} = F\alpha_{(x, y)}$ and ϱ is a well defined, full and dense functor.

Obviously, $\pi : \tilde{Q} \to Q$ such that $(x \to y) \to (Fx \to Fy)$ is a quiver morphism with extension $k\pi : k\tilde{Q} \to kQ$ satisfying $Fr = \varrho k\pi$. $\square$

The morphisms r and ϱ which we have just defined, produce admissible ideals I and $\tilde{I}$ such that $\mathscr{C} \cong k(\tilde{Q}, \tilde{I})$ and $\mathscr{D} \cong k(Q, I)$. In this way $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ is a quiver morphism which preserves relations and $F = k(\pi)$ is the functor induced by π.

3.5. Proposition. *With the same hypothesis and notation as* 3.4, *if $\mathscr{G}(\pi)$ denotes the automorphism group of $(\tilde{Q}, \tilde{I})$ which preserves π, then*

(a) $\mathscr{G}(F) \cong \mathscr{G}(\pi)$.

(b) $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ *is a covering map defined by the action of $\mathscr{G}(F)$.*

Proof. (a) Let $g \in \mathscr{G}(\pi)$. Then there is a unique induced functor $\phi(g)$ such that the following diagram commutes.

Obviously, $\phi(g)$ is an automorphism of $\mathscr{C}$. As $F\phi(g)r = Fr$, then $F\phi(g) = F$ and $\phi(g) \in \mathscr{G}(F)$. In the same way ϕ is a group morphism.

Take now an arbitrary $\bar{g} \in \mathscr{G}(F)$. Observe that if $x \to y$ is an arrow in $\tilde{Q}$, there is exactly one arrow $\bar{g}x \to \bar{g}y$ in $\tilde{Q}$. So $g : \tilde{Q} \to \tilde{Q}$ with $g(x \to y) = \bar{g}x \to \bar{g}y$ is a well-defined quiver morphism. It is easy to prove that g preserves the relations, so that $g : (\tilde{Q}, \tilde{I}) \to (\tilde{Q}, \tilde{I})$ is a morphism of quiver with relations. Besides, for $x \xrightarrow{l} y$ an arrow in $\tilde{Q}$,

$$\pi g(l) = \pi(\bar{g}x \to \bar{g}y) = F\bar{g}x \to F\bar{g}y = Fx \to Fy = \pi(l).$$

So $g \in \mathscr{G}(\pi)$. As $rkg = \bar{g}r$, then $\bar{g} = \phi(g)$. If there is another $g' \in \mathscr{G}(\pi)$ with $\bar{g} = \phi(g')$, g and g' must be equal on arrows, so $g = g'$ and ϕ is an isomorphism.

(b) As $\mathscr{G}(F)$ defines F on objects, it is clear that $\mathscr{G}(F)$ defines the quiver morphism $\pi : \tilde{Q} \to Q$. We also know $\pi(\tilde{I}) \subset I$, so to prove π is a covering map we only need to show $I \subset \pi(\tilde{I})$. Let $\sum_{i=1}^{n} \lambda_i u_i \in I(x, y)$ be a relation. We choose $\tilde{x} \in Q_0$ with $\pi\tilde{x} = x$. As π is defined by the action of a group, it has the property of unique path lifting. So, there is a directed path v_i from $\tilde{x}$ to $\bar{y}_i$ with $\pi v_i = u_i$, $i = 1, \ldots, n$. Without loss of generality we may assume $1 = n_0 < n_1 < \cdots < n_t = n + 1$. So that for any two $j, j' \in \{n_i, \ldots, n_{i+1} - 1\}$, $\bar{y}_j = \bar{y}_{j'}$ and $\bar{y}_i$'s are different in the intervals.

We set

$$\phi_{\tilde{x}_n} = \sum_{i=n}^{n_{i+1}-1} \lambda_i r(v_i) \in k(\tilde{Q}, \tilde{I})(\tilde{x}, \bar{y}_{n_i})$$

for each $j = 1, \ldots, t$. Therefore,

$$\sum_{j=1}^{t} F(\phi_{\bar{y}_n}) = \sum_{i=1}^{n} \lambda_i Fr(v_i) = \sum_{i=1}^{n} \lambda_i \varrho \pi(v_i) = \varrho \left(\sum_{i=1}^{n} \lambda_i u_i \right) = 0.$$

As F is a covering functor, $\phi_{\bar{y}_n} = 0$ and

$$\sum_{n}^{n} \lambda_i v_i \in I(\bar{x}, \bar{y}_{n_j}), \quad j = 1, \ldots, t. \quad \square$$

Let us apply what we have developed up to here to a particular but important case.

3.6. Definition [2]. Let A be a locally finite-representation k-category and Γ its Auslander-Reiten quiver. A is called standard if and only if $\mathrm{ind}\, A \cong k(\Gamma)$.

Let A be a standard category and Γ its Auslander-Reiten quiver. Let $p: \tilde{\Gamma} \to \Gamma$ be the universal covering map. This induces a covering functor $F = k(p): k(\tilde{\Gamma}) \to k(\Gamma)$ and $A \hookrightarrow \mathrm{ind}\, A \cong k(\Gamma)$. Let $\prime$ be the full subcategory of $k(\tilde{\Gamma})$ such that $x \in \mathrm{Ob}\, \prime$ if and only if $px \in A$.

So the situation is:

where $\bar{F}$ denotes the restriction of F. Clearly, $\bar{F}$ is also a covering functor. To apply 3.5 we only need to prove:

3.7. Lemma. $\prime(F)$ acts transitively on fibres of objects.

Proof. Let $x, y \in \mathrm{Ob}\, \prime$ with $\bar{F}x = \bar{F}y$. Then $Fx = Fy$. By definition of the universal cover, there is an automorphism $g \in \prime(p)$ with $gx = y$. Let $\bar{g}: \prime \to \prime$ be the restriction of $k(g): k(\tilde{\Gamma}) \to k(\tilde{\Gamma})$ to $\prime$. Clearly, $\bar{g}$ is a $\prime$-automorphism with $\bar{g} \in \prime(\bar{F})$ and $\bar{g}x = y$.

Let Q be the quiver of A and $\bar{Q}$ the quiver of $\prime$. By 3.7 and 3.5, there are ideals I of Q and $\bar{I}$ of $\bar{Q}$ and $\pi: (\bar{Q}, \bar{I}) \to (Q, I)$ a covering map defined by the action of $\prime(\bar{F})$ such that the following diagram commutes.

This is precisely Gabriel's construction in the standard case; the main result of this section is in the next theorem.

3.8. Theorem. *With the notations introduced above.* $\pi : (\bar{Q}, \bar{I}) \rightarrow (Q, I)$ *is the universal cover.* $\bar{Q}$ *has no oriented cycles.*

Proof. Let $\tilde{\pi} : (\tilde{Q}, \tilde{I}) \rightarrow (Q, I)$ be the universal cover. Then, there is a covering map $\pi' : (\tilde{Q}, \tilde{I}) \rightarrow (\bar{Q}, \bar{I})$ such that $\pi \pi' = \tilde{\pi}$. Assume π' is defined by the action of the group H. We shall prove H is trivial. As (Q, I) is locally representation finite, $(\bar{Q}, \bar{I})$ is also so. Let $\Gamma_{\bar{1}}$ be the Auslander-Reiten quiver of $\bar{A} = k(\bar{Q}, \bar{I})$ and $\Gamma_{\tilde{1}}$ the one of $\tilde{A} = k(\tilde{Q}, \tilde{I})$.

Using [7] and [4], the pushdown functor Σ induces a covering map of translation quivers $\Sigma : \Gamma_{\tilde{1}} \rightarrow \Gamma_{\bar{1}}$ defined by the action of H. As $k(\Gamma) \cong \text{ind} \, A$ is an Auslander category, by [2] $k(\tilde{\Gamma})$ is also so. And by [4] it is the category of indecomposable modules of its projective vertices, which is precisely $\tilde{A}$. So, $k(\tilde{\Gamma}) \cong \text{ind} \, \tilde{A}$ and the Auslander-Reiten quiver of $\tilde{A}$ is $\tilde{\Gamma} = \Gamma_{\tilde{1}}$. But then Σ is the identity and H is trivial. Finally, as $\tilde{\Gamma}$ has no oriented cycles and the inclusion $k(\bar{Q}, \bar{I}) \rightarrow k(\tilde{\Gamma})$ is faithful, $\bar{Q}$ has neither oriented cycles. $\square$

3.9. Corollary. *Let* Γ *be a locally representation finite* k *category. The following conditions are equivalent:*

(i) A *is standard.*

(ii) $A = k(Q, I)$ *and the universal cover* $(\bar{Q}, \bar{I})$ *of* (Q, I) *has no oriented cycles.*

(iii) $A = k(Q, I)$ *with* (Q, I) *satisfying conditions* (C) *and* (D).

Proof. (i) $\Rightarrow$ (ii) is 3.8 (ii) $\Rightarrow$ (iii) is 2.3.
(iii) $\Rightarrow$ (i) follows easily from [3]. $\square$

4. Schurian and standard categories

In this last section we obtain some consequences of the constructions and results of the previous sections.

4.1. Theorem [3]. *Every locally representation finite Schurian category is standard.*

What we do first is to characterize the simply connected categories in the sense of [4], by means of the simple connectedness of the ordinary quiver.

4.2. Theorem. *Let* A *be a locally representation finite category. The following are equivalent:*

(i) A *is simply connected.*

(ii) *There exists a quiver with relations* (Q, I), *with* $A \cong k(Q, I)$ *such that* Q *has no oriented cycles and* (Q, I) *is its own universal cover.*

(iii) *For every quiver with relations (Q, I) with $A \cong k(Q, I)$, Q has no oriented cycles and (Q, I) is its own universal cover.*

Proof. (i) $\Rightarrow$ (ii). By [4], A is standard. By 3.8 (Q, I) is its own universal cover.

(ii) $\Rightarrow$ (iii). Assume $A \cong k(Q, I')$. By 2.6, $\Pi_1(Q, I) = \Pi_1(Q, I')$ is trivial and (Q, I') is its own universal cover.

(iii) $\Rightarrow$ (i). By 2.2, A is Schurian. So by 4.1, A is standard. Then we can apply again 3.8 and observe that $\Pi_1(\Gamma_1)$ acts on the universal cover of (Q, I) which is trivial. So $\Pi_1(\Gamma_1)$ is trivial and A is simply connected. $\square$

We obtain the following corollary.

4.3. Theorem. *Let (Q, I) be a l.r.f. quiver with relations, and $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ its universal cover. Assume $\tilde{Q}$ has no oriented cycles, then $\Pi_1(Q, I) = \Pi_1(\Gamma_1)$ where $A = k(Q, I)$. Hence, $\Pi_1(Q, I)$ is a free group.*

Proof. By [7] the push down functor induces a covering quiver morphism $\Sigma : \Gamma_1 \to \Gamma_1$ defined by $\Pi_1(Q, I)$, where $\tilde{A} = k(\tilde{Q}, \tilde{I})$ and $A = k(Q, I)$. By 4.2, $\tilde{A}$ is simply connected, this implies $\Gamma_{\tilde{I}} = \tilde{\Gamma}_{\tilde{I}} = \tilde{\Gamma}_{1}$. So Σ must be defined by $\Pi_1(\Gamma_1)$ and then $\Pi_1(Q, I) = \Pi_1(\Gamma_1)$ which is free. $\square$

The condition on (Q, I) of being a locally representation finite category is necessary. For example, consider

$$Q: \quad \alpha_1 \circlearrowleft \cdot \underset{\beta_2}{\overset{\beta_1}{\rightleftarrows}} \cdot \circlearrowright \alpha_2$$
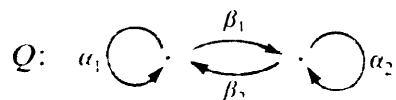
with I generated by

$$\beta_2\beta_1 = 0 = \alpha_1^2 = \alpha_2^2, \qquad \alpha_2\beta_1 = \beta_1\alpha_1, \qquad \beta_2\alpha_2 = \alpha_1\beta_2.$$

(Q, I) has the property of having a universal cover without oriented cycles, but $\Pi_1(Q, I) = \mathbb{Z} \times \mathbb{Z}$ which is not free.

The next results relate the Schurian condition with the fact that the universal cover has no oriented cycles.

4.4. Proposition. *Let (Q, I) be a l.r.f. quiver with relations and $\pi : (\tilde{Q}, \tilde{I}) \to (Q, I)$ its universal cover. Then $\tilde{Q}$ has no oriented cycles if and only if $k(\tilde{Q}, \tilde{I})$ is Schurian.*

Proof. If $\tilde{Q}$ has no oriented cycles, the result follows from 2.1 and 2.2. Assume $k(\tilde{Q}, \tilde{I})$ is Schurian; by 4.1 it is standard. Then by 4.8 we have an ideal $\tilde{I}'$ such that $k(\tilde{Q}, \tilde{I}) = k(\tilde{Q}, \tilde{I}')$ and the universal cover $\bar{\pi} : (\bar{Q}, \bar{I}) \to (\tilde{Q}, \tilde{I})$ has no oriented cycles.

By 2.3, $(\bar{Q}, \bar{I})$ satisfies (C) and (D). But $(\tilde{Q}, \tilde{I})$ is its own universal cover and since $k(\tilde{Q}, \tilde{I})$ is Schurian, $(\tilde{Q}, \tilde{I})$ also satisfies (C) and (D). Then by 2.6, $\bar{Q} = \tilde{Q}$ without oriented cycles.

Using 4.3 and 4.4 we can prove that standard algebras can be constructed begining with Schurian algebras.

4.5. Theorem. *For a finite-dimensional basic and indecomposable k-algebra Λ of finite representation type, the following conditions are equivalent:*

(i) Λ *is standard.*

(ii) *There is a finite-dimensional Schurian algebra $\bar{\Lambda}$ of finite representation type and a covering morphism $p: \bar{\Lambda} \to \Lambda$, that is, there are quivers with relations associated to $\bar{\Lambda}$ and Λ with a covering map between them.*

Proof. If $p: (\bar{Q}, \bar{I}) \to (Q, I)$ is a covering with $k(\bar{Q}, \bar{I})$ Schurian, the common universal cover $(\tilde{Q}, \tilde{I})$ must be Schurian. By 4.4, $\tilde{Q}$ has no oriented cycles. Assume now $\pi: (\tilde{Q}, \tilde{I}) \to (Q, I)$ is the universal cover with $\Lambda \cong k(Q, I)$ and $\tilde{Q}$ has no oriented cycles. π is defined by the action of a free group G, by 4.3. We denote $\bar{\Lambda} = k(\bar{Q}, \bar{I})$ which is Schurian. We proceed now as in 5.2 of [4]: for each $x \in Q_0$ we fix $\tilde{x} \in \tilde{Q}_0$ with $\pi \tilde{x} = x$.

$R_x := \{ \tilde{y} \in \tilde{Q}_0 \mid \mathrm{Hom}_{\bar{A}}(\tilde{x}, \tilde{y}) \neq 0 \}$ is a finite set and as G acts freely on $\tilde{Q}_0$, $S := \{ \gamma \in G \setminus \{1\} \mid \exists x \in Q_0 \text{ with } R_x \cap \gamma(R_x) \neq \emptyset \}$ is finite. As G is free, G is residually finite, so there is a finite index subgroup $P \triangleleft G$ with $P \cap S = \emptyset$. We have
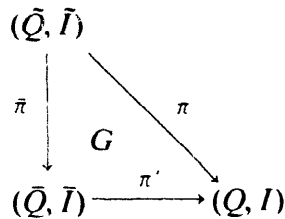
with $\bar{\pi}$ defined by the action of P and π' by that of G/P. $\bar{\Lambda} = k(\bar{Q}, \bar{I})$ is a finite-dimensional algebra of finite representation type. We prove $\bar{\Lambda}$ is Schurian.

Let $s, t \in \bar{Q}_0$ with $\mathrm{Hom}_{\bar{A}}(s, t) \neq 0$. We take $\bar{s} \in \tilde{Q}_0$ with $\bar{\pi}(\bar{s}) = s$, and $\bar{\pi}(\bar{t}) = t$ with $\mathrm{Hom}_{\bar{A}}(\bar{s}, \bar{t}) \neq 0$. There exists $\gamma \in G$, $x \in Q_0$ satisfying $\gamma \bar{s} = \tilde{x}$. Suppose $\bar{\pi} \bar{t}' = t$ and $\mathrm{Hom}_{\bar{A}}(\bar{s}, \bar{t}') \neq 0$, $\bar{t} \neq \bar{t}'$. In this case, we have $1 \neq \delta \in P$ with $\delta \bar{t} = \bar{t}'$. As $0 \neq \mathrm{Hom}_{\bar{A}}(\tilde{x}, \tilde{t})$, $\gamma \bar{t} \in R_x$ and similarly $\gamma \delta \bar{t} = \gamma \bar{t}' \in R_x$. Then $1 \neq \gamma \delta \gamma^{-1}$ such that $\gamma \delta \gamma^{-1} \in P$, $\gamma \delta \gamma^{-1}(\gamma \bar{t}) = \gamma \bar{t}' \in R_x$. Then $\gamma \delta \gamma^{-1} \in S$, and $P \cap S \neq \emptyset$, which is a contradiction. So, $\bar{t} = \bar{t}'$ and

$$\mathrm{Hom}_{\bar{A}}(s, t) = \bigoplus_{\bar{\pi} \bar{t} = t} \mathrm{Hom}_{\bar{A}}(\bar{s}, \bar{t}') = \mathrm{Hom}_{\bar{A}}(\bar{s}, \bar{t})$$

has dimension 1 over k. So $\bar{\Lambda}$ is Schurian. $\square$

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